

BASIC EQUATIONS OF THE METHOD OF PHOTOVISCOELASTICITY

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Nonlinear nonisothermal integral equations are constructed, connecting the deformation and the polarization and optical quantities with the stress. The possibility of applying optical properties to the solution of isothermal and nonisothermal problems of the theory of heredity, using the method of photoviscoelasticity, is evaluated. The piezo-optical effect in polymeric materials is due simultaneously to stress and to deformation, acting as independent parameters. To express the polarization and optical properties only in terms of the stress (or only in terms of the deformation), we must use one or another law of the mechanics of polymers (Hooke's law with real or complex moduli, linear or nonlinear integral equations, etc.). The choice of the rheological law is dictated by the conditions of the deformation. The linear optical equations are well known. In [1-4] it was postulated that they contain operators of a rheological type. In [5], an explanation is given of these equations, starting from the theory of electrodynamics, the theory of the piezo-optical effect, and the linear theory of viscoelasticity. With large stresses, the nonlinear theory of viscoelasticity must be used. Thus, in [6], the isothermal theory of ageing was used, bearing in mind the problems of quasi-established viscoelasticity. We obtain more general equations using the nonlinear heredity theory and the principle of temperature-time correspondence.

1. Starting Optical Equations. We limit ourselves to the plane problem of electrodynamics. Let a layer of a nonmagnetic dielectric be penetrated without losses by monochromatic electromagnetic waves normal to the plane of the layer x_1x_2 ; the material is originally optically isotropic and homogeneous; n_0 is the refractive index of the material. If the dielectric permeability varies only slightly with deformation of the layer, the optical path difference δ , referred to the thickness of the layer, and the isocline φ , are connected with the stresses (σ_{ij}) and the deformation (ε_{ij}) by the relationships [5, 7]

$$\delta \cos 2\varphi = C_\sigma (\sigma_{11} - \sigma_{22}) + C_\varepsilon (\varepsilon_{11} - \varepsilon_{22}), \quad 1/2 \delta \sin 2\varphi = C_\sigma \sigma_{12} + C_\varepsilon \varepsilon_{12} \quad (1.1)$$

Here C_σ and C_ε are coefficients which, in the linear theory of the piezo-optical effect, are independent of the mechanical properties and, for stable materials, also of the time. The deformations are assumed to be small.

It must be emphasized that the values of δ and φ are determined by the form and the orientation of a section of the dielectrical ellipsoid by the plane of the wave front; therefore, a sharp interference picture of the bands and of the isoclines can be obtained not only with elastic but with nonelastic deformation of the layer.

We transform to the principal stresses and deformations in the plane of the layer; then, after simple transformations, we obtain expressions for δ and φ , which make it possible to interpret the interference picture of the bands and isoclines

$$\delta^2 = C_\sigma^2 (\sigma_1 - \sigma_2)^2 + C_\varepsilon^2 (\varepsilon_1 - \varepsilon_2)^2 + 2C_\sigma C_\varepsilon (\sigma_1 - \sigma_2) (\varepsilon_1 - \varepsilon_2) \cos 2(\varphi_\sigma - \varphi_\varepsilon)$$

$$\operatorname{tg} 2\varphi = \frac{C_\sigma (\sigma_1 - \sigma_2) \sin 2\varphi_\sigma + C_\varepsilon (\varepsilon_1 - \varepsilon_2) \sin 2\varphi_\varepsilon}{C_\sigma (\sigma_1 - \sigma_2) \cos 2\varphi_\sigma + C_\varepsilon (\varepsilon_1 - \varepsilon_2) \cos 2\varphi_\varepsilon} \quad \begin{matrix} (\sigma_1 \geq \sigma_2) \\ (\varepsilon_1 \geq \varepsilon_2) \end{matrix}$$

Here, φ_σ and φ_ε are the mechanical isoclines. It can be seen that, in general, the bands are not determined by the differences $\sigma_1 - \sigma_2$ or $\varepsilon_1 - \varepsilon_2$, and that the optical isoclines do not coincide with the mechanical. This situation will obtain only in an isotropic layer (photoelasticity).

Leningrad. Translated from *Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki*, No. 2, pp. 72-75, March-April, 1971. Original article submitted September 30, 1970.

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Let Hooke's law not be satisfied, and let the prehistory of the deformation be arbitrary. The following cases are encountered. When $C_\varepsilon = 0$, the bands are determined by the difference of the principal stresses, and the optical isoclines by the directions of the latter. If $C_\sigma = 0$, the bands depend only on the difference of the principle elongations, and the optical isoclines on the directions of ε_1 and ε_2 . Finally, when the principal axes (σ_{ij}) , (ε_{ij}) in a mechanically isotropic layer are fixed ($\text{tg } 2\varphi_\sigma = \text{tg } 2\varphi_\varepsilon$), the optical isoclines coincide with the mechanical, and the Fylen Jesson law, connecting δ with $\sigma_1 - \sigma_2$ and $\varepsilon_1 - \varepsilon_2$, is satisfied. In what follows, we shall assume that $C_\sigma \neq 0$, and $C_\varepsilon \neq 0$.

2. Rheological Equations. In [8-10] it was proposed to use the hypothesis of thermoheologically simple behavior in the nonlinear theory of viscoelasticity; in [11], this hypothesis was verified with large deformations. Thermoheologically simple behavior is described using the reduced time, ξ , which coincides with the true time only at some temperature T_0 chosen as the starting temperature (the reference temperature)

$$\xi(t) = \int_0^t g [T(\eta)] d\eta \quad (g(T) > 0, \partial g(T) / \partial T > 0, g(T_0) = 1)$$

The proportionality coefficient, g , plays a role depending on the temperature scale of the time. We assume that g does not depend on the prehistory of the deformation, the level of the stresses, or the type of stressed state.

In accordance with the hypothesis of thermoheologically simple behavior, the equations for the viscoelasticity of mechanically isotropic materials under constant stresses [12] are described in the form

$$\begin{aligned} 2\partial_{ij}(t) &= L(\xi, s) s_{ij}, & \frac{1}{3}\Delta &= K\sigma_m + \varepsilon^T \\ L(0, s) &= 1/G, & s &= +\sqrt{\frac{1}{2}s_{ij}s_{ij}}, \quad \sigma_m = \frac{1}{3}\sigma_{ii}, \quad \Delta = \varepsilon_{ii} \quad (\varphi_\sigma = \varphi_\varepsilon) \end{aligned} \quad (2.1)$$

Here s is the intensity of the tangential stresses; σ_m is the mean pressure; Δ is the relative change in the volume; ε^T is the thermal expansion; G is the shear modulus; K is the modulus of the hydrostatic pressure; $L(\xi, s)$ is a positive increasing function of its arguments. It is assumed that the volume varies in accordance with an elastic law. If $L(\xi, s)$ is replaced by a function of the time $l(\xi)$, we obtain the well-known linear equations.

Using (2.1) and a modified summation rule [12], we obtain the equations of the nonlinear nonisothermal heredity theory

$$\begin{aligned} 2\partial_{ij}(t) &= L(0, s) s_{ij}(t) - \int_0^t L'_\omega [\xi - \zeta, s(\omega)] s_{ij}(\omega) d\omega \\ \frac{1}{3}\Delta &= K\sigma_m + \varepsilon^T, \quad \zeta = \xi(\omega), \quad \xi - \zeta = \int_\omega^t g [T(\eta)] d\eta, \quad \xi - \zeta = t - \omega \quad (T = T_0) \end{aligned} \quad (2.2)$$

where L'_ω is the partial derivative with respect to the integration variable ω , entering into the first argument of the function $L[\xi - \zeta, s(\omega)]$.

3. Principal Optical Equations. We substitute (2.1) into (1.1) and transform to the principal axes; then, for constant stresses, we obtain

$$\pm \delta(t) = C(\xi, s) (\sigma_1 - \sigma_2), \quad \text{tg } 2\varphi = \text{tg } 2\varphi_\sigma \quad (3.1)$$

$$C(\xi, s) = C_\sigma + \frac{1}{2} C_\varepsilon L(\xi, s), \quad C(0, s) = C_\sigma + \frac{1}{2} C_\varepsilon / G \quad (3.2)$$

The function $C(\xi, s)$ may be either positive or negative. For example,

$$C(\xi, s) < 0 \quad \text{for } C_\sigma > 0, C_\varepsilon < 0, C_\sigma < \left| \frac{1}{2} C_\varepsilon L(\xi, s) \right|$$

With stresses varying with time, Eqs. (2.2), and not (2.1), are used

$$\begin{aligned} \delta(t) \cos 2\varphi(t) &= C(0, s) [\sigma_{11}(t) - \sigma_{22}(t)] - \int_0^t C_{\omega}' [\xi - \zeta, s(\omega)] [\sigma_{11}(\omega) - \sigma_{22}(\omega)] d\omega \\ \frac{1}{2} \delta(t) \sin 2\varphi(t) &= C(0, s) \sigma_{12}(t) - \int_0^t C_{\omega}' [\xi - \zeta, s(\omega)] \sigma_{12}(\omega) d\omega \end{aligned} \quad (3.3)$$

Within the framework of the theory of the piezo-optical effect, the optical and rheological operators are, in accordance with (3.2), interconnected by a very simple relationship. If the piezo-optical coefficients C_{σ} and C_{ε} do not depend on the temperature, as is true, for example, for celluloid or epoxy resin, not only the mechanical but also the optical properties will be thermoelastically simple, and the temperature scales of the time will be found to be identical for both mechanical and optical quantities.

4. Monoaxial Elongation-Compression. We designate the component of the stress which differs from zero, and the corresponding component of the deformation, by σ and ε . In accordance with (1.1) and (2.1), we have

$$\begin{aligned} \delta &= C_1 \sigma + C_2 \varepsilon \quad (\varphi = 0), & -\delta &= C_1 \sigma + C_2 \varepsilon \quad (\varphi = 90^\circ) \\ e &= \varepsilon - \varepsilon^T, & C_1 &= C_{\sigma} - \frac{1}{2} C_{\varepsilon} K, & C_2 &= \frac{3}{2} C_{\varepsilon} \end{aligned} \quad (4.1)$$

In accordance with (2.1), (3.1), and (4.1), at $\sigma = \text{const}$, we obtain (see [12])

$$e(t) = \frac{1}{3} [L(\xi, \frac{1}{3} \sqrt{3} |\sigma|) + K] \sigma = D(\xi, \sigma) \quad (4.2)$$

$$s = \frac{1}{3} \sqrt{3} |\sigma|, \quad D(0, \sigma) = \sigma / E$$

$$\pm \delta(t) = C(\xi, \frac{1}{3} \sqrt{3} |\sigma|) \sigma = J(\xi, \sigma) \quad (4.3)$$

$$J(\xi, \sigma) = C_1 \sigma + C_2 D(\xi, \sigma), \quad J(0, \sigma) = (C_1 + C_2 / E) \sigma \quad (4.4)$$

Here, E is the Young modulus. In accordance with (4.2) and (4.3), the mechanical and optical properties with elongation-compression are assumed to be identical. In the case of a stress which varies in time

$$\begin{aligned} e(t) &= D[0, \sigma(t)] - \int_0^t D_{\omega}' [\xi - \zeta, \sigma(\omega)] d\omega \\ \pm \delta(t) &= J[0, \sigma(t)] - \int_0^t J_{\omega}' [\xi - \zeta, \sigma(\omega)] d\omega \end{aligned} \quad (4.5)$$

5. Application of (3.3) in Photomechanics. The optical equations play in photomechanics the same role as the determining equations in the mechanics of deformed bodies; (3.3) are the principal equations of the method of photoviscoelasticity.

The mechanical and optical characteristics of the materials of models can be found, for example, from experiments on samples at constant loads and temperatures; measurements must be made of $e(t)$ and $\delta(t)$ at successive moments of time. Using a plot in the coordinates $e(t)/\sigma$, $\delta(t)/\sigma$, the coefficients C_1 , C_2 , and C_{ε} are found, the degree of their temperature dependence is established, and the region of the linear piezo-optical effect is determined. If the modulus K is known, C_{σ} may also be calculated. A comparison of the curves for $e(t)/\sigma$ and $\delta(t)/\sigma$ on a logarithmic time scale makes it possible to plot a separate superimposed curve for each stress, to determine $g(T)$, and to find the region of nonlinear viscoelasticity for each temperature. In accordance with (4.2) the curves of $e(t)$ define the mechanical functions $D(\xi, s)$ and $L(\xi, s)$, while, in accordance with (4.3), the curves of $\delta(t)$ define the optical functions $J(\xi, s)$ and $C(\xi, s)$ at $s = \frac{1}{3} \sqrt{3} |\sigma|$. Another method for obtaining the optical functions is by the use of the piezo-optical coefficients and the mechanical functions, in accordance with (3.2) and (4.4).

After the functions $C(\xi, s)$ and $g(T)$ have been defined, Eqs. (3.3) can be applied to the solution of plane problems of the nonlinear heredity theory of viscoelasticity, independently of whether or not there is any considerable amount of thermomechanical interaction. In the experiments, $\delta(t)$, $\varphi(t)$, and $T(t)$ must be recorded. In the calculations, one of the approximate methods is used, and the functions C and g can be found in tables.

In the case of the investigation of stresses at free contours, with a plane state of stress, the problem reduces to the numerical solution of a nonlinear integral Volterra equation of the second kind (4.5). Investigations within the region are more complex. As a rule, $\delta(t)$, $\varphi(t)$, and $T(t)$ must be measured not only in the cross section being investigated, but also in one or two other closely located auxiliary sections. In the case of quasi-statistical problems, Eqs. (3.3) are supplemented in the calculation by a differential equilibrium equation. Since the function C depends on s , the problem cannot be solved in two steps (as in the methods of photoelasticity and linear photoviscoelasticity); at first $\sigma_{11} - \sigma_{22}$ and σ_{12} are found using the optical equations, and then, using the differential equilibrium equation, σ_{11} and σ_{22} are found. The system of three equations must be solved simultaneously.

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